Energy Scale of the Big Bounce

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We examine the nature of the cosmological big bounce (BB) transition within the loop geometry underlying Loop Quantum Cosmology (LQC) at classical and quantum levels. Our canonical quantization method is an alternative to the standard LQC. An evolution parameter we use has a clear interpretation. Our method opens door for analyzes of spectra of physical observables like the energy density and the volume operator. We find that one cannot determine the energy scale specific to BB by making use of the loop geometry without an extra input from observational cosmology.

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1. Introduction. It is commonly believed that the unification of gravity and quantum mechanics would solve the intriguing problem of singularities in general relativity (GR). Canonical quantization is the preferred method of quantization of GR. In the canonical formulation of GR the Hamiltonian is a sum of first-class constraints. There are two ways of quantization of such systems [1, 2] based on prescriptions: (i) 'first quantize kinematics, then impose constraints', or (ii) 'first solve classical constraints, then impose quantum rules'. The latter case is applied in the present paper. The former one is called the Dirac quantization and it is the quantization method used in the so called Loop Quantum Gravity (LQG) (see, e.g. [3, 4]) and Loop Quantum Cosmology (LQC) (see, e.g. [5, 6]).

LQC deals with the simplest cosmological models of the universe. It offers the resolution of the initial big-bang singularity in the sense that the singularity is replaced by the regular big-bounce (BB) transition (see, e.g. [7, 8, 9, 10]). Revealing the nature of the initial singularity is a prerequisite for understanding the origin of matter, non-gravitational fields and spacetime. Thus, much importance has been ascribed to this result.

We define the volume operator and present preliminary results concerning its spectrum in the physical Hilbert space.

The aim of our paper is to show that the energy scale specific to the BB transition cannot be determined uniquely because the energy density of matter specific to BB depends on a free parameter labelling the loop geometry. The determination of this scale is of primary importance as it would help to identify the unification scale of gravity with quantum physics.

2. Classical dynamics. For simplicity of exposition we restrict ourselves to the quantization problem of flat Friedmann-Robertson-Walker (FRW) model with massless scalar field. The metric in this model reads

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\,(dx^2 + dy^2 + dz^2), \tag{1}$$

where a is the scale factor and N denotes the lapse function. In this simple cosmological set-up the classical dynamics is defined by the Hamiltonian:

$$H = N\left(-\frac{3}{8\pi G\gamma^2}\,\beta^2\,v + \frac{p_\phi^2}{2\,v}\right), \tag{2}$$

where (β, v, ϕ, p_ϕ) are the kinematical phase space variables and γ is the so-called Barbero-Immirzi parameter. In terms of

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scale factor one has $\beta = \gamma\dot{a}/|a|$, $v = a^3$, and p_ϕ is the momentum of the field ϕ. This model of the universe unavoidably includes the initial cosmological singularity and has been intensively studied recently within LQC.

Loop geometry is based on the use of a canonical pair 'holonomy and flux' instead of a canonical pair 'connection and triad' [5, 6]. The mapping from 'connection' to 'holonomy' is not invertible. The holonomy function $h_k^{(\lambda)}$ (along straight line of coordinate length proportional to $\lambda/|a|$) in the fundamental, j = 1/2, representation of SU(2) group reads [9]

$$h_k^{(\lambda)}(\beta) = \cos(\lambda\beta/2)\,\mathbb{I} + 2\,\sin(\lambda\beta/2)\,\tau_k, \quad (3)$$

where $\tau_k = -i\sigma_k/2$ (σ_k are the Pauli spin matrices). It transforms the gravitational part of the Hamiltonian of the FRW model into [9]

$$H_g = \lim_{\lambda \to 0} H_g^{(\lambda)}, \quad (4)$$

where

$$H_g^{(\lambda)} = -\frac{v}{2\pi G\gamma^3\lambda^3}\sum_{ijk} N\,\varepsilon^{ijk}\,Tr\Big(h_i^{(\lambda)}h_j^{(\lambda)}$$
$$\times (h_i^{(\lambda)})^{-1}(h_j^{(\lambda)})^{-1}h_k^{(\lambda)}\{(h_k^{(\lambda)})^{-1}, v\}\Big), (5)$$

and where ε^{ijk} is the alternating tensor. The Poisson bracket is defined to be

$$\{\cdot,\cdot\} := 4\pi G\gamma\left[\frac{\partial\cdot}{\partial\beta}\frac{\partial\cdot}{\partial v} - \frac{\partial\cdot}{\partial v}\frac{\partial\cdot}{\partial\beta}\right]$$
$$+\frac{\partial\cdot}{\partial\phi}\frac{\partial\cdot}{\partial p_\phi} - \frac{\partial\cdot}{\partial p_\phi}\frac{\partial\cdot}{\partial\phi}. \quad (6)$$

Roughly speaking, quantization of $H^{(\lambda)} := H_g^{(\lambda)} + H_\phi$ (where H_ϕ is the matter part of the Hamiltonian) for fixed value of λ is the essence of LQC [7, 8, 9, 10]. It means that H_g has been approximated by $H_g^{(\lambda)}$, since λ is assumed to be different from zero. First, one quantizes the whole phase space, i.e. assigns quantum operators to the variables (β, v, ϕ, p_ϕ) which act in the kinematical Hilbert space (KHS) [11, 12]. Next, one

solves the operator equation

$$\widehat{H}^{(\lambda)}\psi = 0, \quad (7)$$

corresponding to the classical constraint equation $H^{(\lambda)} = 0$. The task of solving the above equation is far from trivial. Usually, the space of solutions, $\mathcal{F}$, is not contained in KHS. To construct the physical Hilbert space (PHS) one uses the dual space to $\mathcal{F}$ and special techniques called the group-averaging method [13, 14]. All that brings about a lot of analytical and numerical work even for the simple cosmological model considered here.

In what follows we present the results obtained by using our quantization method. Making use of (3) in (5) leads directly to the modified total Hamiltonian corresponding to (2) given by

$$H^{(\lambda)} = N\left(-\frac{3}{8\pi G\gamma^2}\frac{\sin^2(\lambda\beta)}{\lambda^2}\,v + \frac{p_\phi^2}{2\,v}\right) \quad (8)$$

The new Hamiltonian is bounded, as a function of β, and leads to a modified, singularity-free classical dynamics. Although, $H^{(\lambda)}$ may generate dynamics in the whole phase space, the physical sector is constrained to the surface $H^{(\lambda)} = 0$.

By an observable we mean a function on the phase space which has vanishing Poisson bracket with the Hamiltonian. We call it an elementary if it cannot be expressed as a non-invertible function of another observable. For simplicity of calculations we fix the gauge by setting

$$N^{-1} := \frac{3}{8\pi G\gamma^2 v}\left(\kappa\gamma|p_\phi| + v\frac{|\sin(\lambda\beta)|}{\lambda}\right), \quad (9)$$

where $\kappa^2 \equiv 4\pi G/3$. Consequently, (8) reduces to

$$H^{(\lambda)} = \kappa\gamma|p_\phi| - v\frac{|\sin(\lambda\beta)|}{\lambda}. \quad (10)$$

To identify all observables of our system, we solve the equation

$$\{\mathcal{O}_j, H^{(\lambda)}\} = 0. \quad (11)$$

We find that all possible functionally independent elementary observables are [15]

$$\mathcal{O}_1 := p_\phi,$$
$$\mathcal{O}_2 := \phi - \frac{\mathrm{sgn}(p_\phi)}{3\kappa}\,\mathrm{arth}\big(\cos(\lambda\beta)\big),$$
$$\mathcal{O}_3 := \mathrm{sgn}(p_\phi)\,v\,\frac{\sin(\lambda\beta)}{\lambda}. \tag{12}$$

They satisfy the Lie algebra

$$\{\mathcal{O}_2,\mathcal{O}_1\} = 1, \quad \{\mathcal{O}_1,\mathcal{O}_3\} = 0,$$
$$\{\mathcal{O}_2,\mathcal{O}_3\} = \gamma\kappa. \tag{13}$$

For $p_\phi = 0$ the algebra (13) is not well defined, but we make an extension of it to include this case. The constraint equation, $H^{(\lambda)} = 0$, takes the simple form

$$\gamma\kappa\,\mathcal{O}_1 = \mathcal{O}_3. \tag{14}$$

Eliminating $\mathcal{O}_3$ from the algebra (13), by using (14), leads finally to a very simple algebra for just two variables

$$\{\mathcal{O}_2,\mathcal{O}_1\} = 1, \tag{15}$$

where $\mathcal{O}_1, \mathcal{O}_2 \in \mathbb{R}$.

3. Energy density and volume operators. Since $\dot\phi := \{\phi, H^{(\lambda)}\} = \kappa\gamma\,\mathrm{sgn}(p_\phi)$ is positive or negative (for $p_\phi \neq 0$), ϕ changes monotonically so it can be used as an evolution parameter. To find an evolution of v in terms of ϕ, we consider the equation

$$\frac{dv}{d\phi} = \frac{\dot v}{\dot\phi} = \frac{\{v, H^{(\lambda)}\}}{\{\phi, H^{(\lambda)}\}}, \tag{16}$$

which in rewritten form is

$$\frac{\mathrm{sgn}(\sin(\lambda\beta))}{\cos(\lambda\beta)}\,\frac{dv}{v} = 3\kappa\,\mathrm{sgn}(p_\phi)\,d\phi. \tag{17}$$

Solution to (17) in terms of elementary observables reads [15]

$$v(\phi) = \kappa\gamma\lambda\,|\mathcal{O}_1|\,\cosh\big(3\kappa(\phi - \mathcal{O}_2)\big). \tag{18}$$

Taking into account that the energy density of the scalar field is given by $\rho = p_\phi^2/2v^2$, we get

$$\rho(\phi) = \frac{1}{2(\kappa\gamma\lambda)^2\cosh^2\big(3\kappa(\phi - \mathcal{O}_2)\big)}. \tag{19}$$

The bounce occurs at the maximum of the energy density

$$\rho_{\mathrm{max}} = \frac{1}{2}\frac{1}{(\kappa\gamma\lambda)^2}. \tag{20}$$

The expressions (18) and (19) show that v and ρ may be interpreted as a family of observables labelled by ϕ. The physical phase space is now parametrized only by $\mathcal{O}_1$ and $\mathcal{O}_2$. The ϕ variable, an evolution parameter, does not belong to the physical phase space. Thus, it will stay classical during the quantization process. Instead of ϕ, we may use any evolution parameter specified by the choice of gauge N in (8). Such a possibility always exists in the case of globally hyperbolic spacetimes. In LQC, contrary to our method, ϕ is a phase space variable so it must be quantized [7, 8, 9, 10]. Being a quantum variable it may fluctuate so its use in LQC as an evolution parameter at the quantum level has poor interpretation.

3. Quantum dynamics. Contrary to the Dirac method, our quantization method of constrained systems is simple enough to be fully controlled analytically. In the Schrödinger representation (since $\mathcal{O}_1, \mathcal{O}_2 \in \mathbb{R}$) we have

$$\mathcal{O}_2 \mapsto \widehat{\mathcal{O}}_2 := x \quad , \quad \mathcal{O}_1 \mapsto \widehat{\mathcal{O}}_1 := -\imath\hbar\partial_x, \tag{21}$$

where $x \in \mathbb{R}$. The representation of (15) defined in the Hilbert space $\mathbb{L}^2(\mathbb{R})$ reads

$$[\widehat{\mathcal{O}}_2, \widehat{\mathcal{O}}_1] = \imath\hbar. \tag{22}$$

In this representation the energy density operator takes the very simple form

$$\rho \to \widehat{\rho} := \frac{1}{2(\kappa\gamma\lambda)^2\cosh^2[3\kappa(\phi - x)]}. \tag{23}$$

Solution to the eigenvalue problem

$$\widehat{\rho}\psi = \rho(x_0)\psi \tag{24}$$

for fixed value of ϕ reads

$$\psi_1 = \delta(x - x_0) \ , \ \psi_2 = \delta(x + x_0 - 2\phi). \tag{25}$$

The eigenvectors (25) are generalized vectors. The spectrum $(0, \frac{1}{2(\kappa\gamma\lambda)^2})$ is doubly degenerate since $cosh(\cdot)$ is a symmetric function.

Since the evolution (parametrized by ϕ) of the eigenvalue corresponding to the generalized eigenvector $\delta(x - x_0)$ reads

$$\widehat{\rho}\,\delta(x - x_0) = \frac{\delta(x - x_0)}{2(\kappa\gamma\lambda)^2 \cosh^2\left(3\kappa(\phi - x_0)\right)}, \tag{26}$$

we conclude that the evolution of the energy density is the same as the classical one (19). It is expected that the gaussian states, approximating generalized vectors, have similar properties.

The resolution of the initial singularity proposed within LQC [5, 6, 7, 8, 9, 10] has been obtained due to the import of the discreteness of the kinematical geometrical operators from LQG to LQC [12]. However, the kinematical discreteness does not necessarily extend to the physical Hilbert space, which lies outside of the kinematical Hilbert space (see comments following (7)). It is straightforward to show that the claim does not hold in the model considered here.

The quantum volume operator corresponding to (18) may be defined as follows [16]

$$\widehat{v} = \kappa\gamma\lambda \frac{1}{2} \left| \widehat{\mathcal{O}}_1 \cosh[3\kappa(\phi - \widehat{\mathcal{O}}_2)] \right.$$
$$\left. + \cosh[3\kappa(\phi - \widehat{\mathcal{O}}_2)]\widehat{\mathcal{O}}_1 \right|$$
$$= \kappa\gamma\lambda\hbar \left| -\frac{3}{2}\kappa \sinh[3\kappa(\phi - x)] \right.$$
$$\left. + \cosh[3\kappa(\phi - x)]\partial_x \right|. \tag{27}$$

It is not difficult to find that the eigenvalue problem for the operator $\widehat{v}$ (for a fixed value of ϕ) has the solution

$$\widehat{v}\psi = |v|\psi, \tag{28}$$

$$\psi = \frac{\sqrt{\frac{3\kappa}{\pi}} \exp\left(i \frac{2v}{3\kappa^2\gamma\lambda\hbar} \arctan e^{3\kappa(\phi - x)}\right)}{\cosh^{\frac{1}{2}}[3\kappa(\phi - x)]}, \tag{29}$$

where $v \in \mathbb{R}$ and $\psi \in \mathbb{L}^2(\mathbb{R})$ is normalized.

The spectrum of the volume operator $\widehat{v}$ appears to be continuous, but our recent analyzes has shown that it is discrete [17].

4. *Conclusions.* The global hyperbolicity of spacetime enables identification of an evolution parameter (e.g. ϕ) at the classical level. In our scheme its use has been extended to the quantum level. Such procedure is possible because we do not quantize the constraint $H^{(\lambda)} = 0$, but the set of observables which does not include ϕ.

The energy scale specific to the Big Bounce can be determined from (20) (quantum and classical energy densities coincide due to (23) and (19)), but it is not unique because λ is a free parameter of the formalism.

The parameter λ has been fixed in LQC by using a discrete spectrum of the kinematical area operator of LQG. It is an assumption of LQC which leads to the commonly expected result that the Big Bounce transition occurs at the Planck scale [7, 8, 9, 10]. It is argued in [12] and [18] that this assumption has poor physical justification. In fact, an association of the Big Bounce with the Planck scale (within loop cosmology) may be done easily. If we fix suitably the value of λ, our results may fit the Planck scale too: substituting $\lambda = l_{Pl}$ into (20) gives $\rho_{max} \simeq 2,07\,\rho_{Pl}$, and taking $\lambda = 1,44\,l_{Pl}$ leads to $\rho_{max} \simeq \rho_{Pl}$ (l_{Pl} and ρ_{Pl} denote the Planck length and energy density, respectively, and we use $\gamma \simeq 0.24$ determined in the black hole entropy calculations [19, 20]). However, real challenge is finding a sound physical justification for the specific choice of λ.

The spectrum of primordial gravitational waves is expected to be sensitive to the holonomy corrections of the loop cosmology (see, e.g., [21, 22, 23]). Detection of the cosmological tensor perturbations may help to determine λ and identify the energy scale of the Big Bounce.

Our quantization method, which we applied to FRW model, may be extended to

other cosmologies including simple homogeneous (e.g. Bianchi I) and isotropic (e.g. Lemaître-Tolman) models. Our next paper will concern the Bianchi I universe.

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